Dynamics in Differential and Difference Algebra

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Integration Problems

Indefinite Integration. Given a function $f(x)$ in certain class $\mathfrak{C}$, decide whether there exists $g(x) \in \mathfrak{C}$ such that

$$f = \frac{dg}{dx} \triangleq g'.$$

Example. For $f = \log(x)$, we have $g = x\log(x) - x$.

Definite Integration. Given a function $f(x)$ that is continuous in the interval $I \subseteq \mathbb{R}$, compute the integral

$$\int_I f(x)dx.$$

Example. For $f = \log(x)$ and $I = [1,2]$, we have

$$\int_I f(x)dx = 2\log(2) - 1.$$

Fundamental Theorem of Calculus

Newton–Leibniz Theorem. Let $f(x)$ be a continuous function on $[a,b]$ and let $F(x)$ be defined by

$$F(x) = \int_a^x f(t)dt \quad \text{for all } x \in [a,b].$$

Then $F(x)' = f(x)$ for all $x \in [a,b]$ and

$$\int_a^b f(x)\,dx = F(b) - F(a). \qquad \text{(Newton–Leibniz formula)}$$

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Definite Integration $\rightsquigarrow$ Indefinite Integration

$$\int_1^2 \log(x)\,dx = F(2) - F(1) = 2\log(2) - 1, \quad \text{where } F(x) = x\log(x) - x.$$

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Definite Integration $\rightsquigarrow$ Indefinite Integration

$$\int_0^{+\infty} \exp(-x^2)\,dx = ?$$

What is Elementary Functions?

$$\mathfrak{E} := (\{\mathbb{C}, x\}, \quad \{+, -, \times, \div\}, \quad \{\exp(\cdot), \log(\cdot), \mathsf{RootOf}(\cdot)\}).$$

Definition. An elementary function is a function of x which is the composition of a finite number of

- binary operations: $+, -, \times, \div$;
- unary operations: exponential, logarithms, constants, solutions of polynomial equations.

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Example.

$$3x^2 + 3x + 1$$

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$$\frac{1}{3x^2 + 3x + 1}$$

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Example.

$$\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1$$

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Example.

$$\log\left(\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1\right)$$

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Example.

$$\frac{\pi}{\sqrt{\log\left(\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1\right)}}$$

Differential Algebra

Differential Ring and Differential Field. Let R be an integral domain. An additive map $D: R \to R$ is called a derivation on R if

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f). \qquad \text{(Leibniz's rule)}$$

The pair (R, D) is called a differential ring. If R is a field, it is then called a differential field.

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Example.
Polynomial ring: $(\mathbb{C}[x], ')$

$$P = \sum_{i=0}^{n} p_i x^i \quad \rightsquigarrow \quad P' = \sum_{i=0}^{n} i p_i x^{i-1}.$$

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Example.
Rational-function field: $(\mathbb{C}(x), ')$

$$f = \frac{P}{Q} \quad \rightsquigarrow \quad f' = \frac{P'Q - PQ'}{Q^2}.$$

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Example.
Elementary-function field: algebraic case

$$(\mathbb{C}(x)(\alpha), ') \text{ with } \alpha \text{ algebraic over } \mathbb{C}(x)$$

$$r_d\alpha^d + r_{d-1}\alpha^{d-1} + \cdots + r_0 = 0 \quad \rightsquigarrow \quad \alpha'(x) = -\frac{r_d'\alpha^d + \cdots + r_0'}{dr_d\alpha^{d-1} + \cdots + r_1}$$

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Example.
Elementary-function field: exponential case

$$(\mathbb{C}(x)(\exp(x)), ')$$

$$f = \frac{1 + x + \exp(x)}{x^2 + \exp(x)} \quad \rightsquigarrow \quad f' = \frac{x(x\exp(x) - 3\exp(x) - x - 2)}{(x^2 + \exp(x))^2}.$$

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Example.
Elementary-function field: logarithmic case

$$(\mathbb{C}(x)(\log(x)),')$$

$$f = \frac{1 + x + \log(x)}{x^2 + \log(x)} \quad \rightsquigarrow \quad f' = -\frac{2\log(x)x^2 + x^3 - \log(x)x + x^2 + x + 1}{(x^2 + \log(x))^2 x}.$$

Differential Algebra

Differential Ring and Differential Field. Let R be an integral domain. An additive map $D: R \to R$ is called a derivation on R if

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f). \qquad \text{(Leibniz's rule)}$$

The pair (R, D) is called a differential ring. If R is a field, it is then called a differential field.

Example.
Elementary-function field: general case

$$(\mathbb{C}(x)(t_1, t_2, t_3, \ldots, t_n), ')$$

$$t_1 = \sqrt{x^2+1}, \quad t_2 = \log(1+t_1^2), \quad t_3 = \exp\left(\frac{1+t_1}{t_1+t_2^2}\right), \ldots$$

Elementary Extensions

Differential Extension. (R^*, D^*) is called a differential extension of (R, D) if $R \subseteq R^*$ and $D^* |_R = D$.

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Elementary Extension. Let (E, D) be a differential extension of (F, D). An element $t \in E$ is elementary over F if one of the following conditions holds:

- t is algebraic over F;
- $D(t)/t = D(u)$ for some $u \in F$, i.e., $t = \exp(u)$;
- $D(t) = D(u)/u$ for some $u \in F$, i.e., $t = \log(u)$.

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Example. $(F, D) = (\mathbb{C}(x), ')$ and $(E, D) = (\mathbb{C}(x, \log(x)), ')$.

Elementary Functions

Definition. An function $f(x)$ is elementary if $\exists$ a differential extension $(E,')$ of $(\mathbb{C}(x),')$ s.t. $E = \mathbb{C}(x)(t_1,\ldots,t_n)$ and t_i is elementary over $\mathbb{C}(x)(t_1,\ldots,t_{i-1})$ for all $i = 2,\ldots,n$.

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Example.

$$f(x) = \frac{\pi}{\sqrt{\log\left(\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1\right)}}$$

Then $f(x)$ is elementary since $\exists$ a differential extension

$$E = \mathbb{C}(x)(t_1,t_2,t_3,t_4),$$

where

$$t_1 = \sqrt{\frac{1}{3x^2+3x+1}}, \quad t_2 = \exp(t_1), \quad t_3 = \log(t_2^2 + x^2 + 1), \quad t_4 = \sqrt{t_3}.$$

Symbolic Integration

Let (F, D) and (E, D) be two differential fields such that $F \subseteq E$.

Problem. Given $f \in F$, decide whether there exists $g \in E$ s.t. $f = D(g)$. If such g exists, we say f is integrable in E.

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Elementary Integration Problem. Given an elementary function $f(x)$ over $\mathbb{C}(x)$, decide whether $\int f(x)dx$ is elementary or not.

Example. The following integrals are not elementary over $\mathbb{C}(x)$:

$$\int \exp(x^2)dx, \quad \int \frac{1}{\log(x)}dx, \quad \int \frac{\sin(x)}{x}dx, \quad \int \frac{dx}{\sqrt{x(x-1)(x-2)}}, \quad \cdots$$

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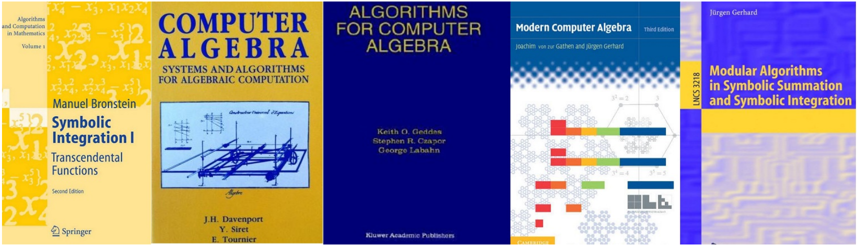
Selected books on Symbolic Integration:



Liouville's Theorem

Theorem (Liouville1835). Let $f(x)$ be elementary over $\mathbb{C}(x)$, i.e.,

$$f \in F = \mathbb{C}(x)(t_1, t_2, \ldots, t_n).$$

If $\int f(x)\,dx$ is elementary, then

$$\int f(x)\,dx = \underbrace{g_0}_{F\text{-part}} + \underbrace{\sum_{i=1}^{n} c_i \log(g_i)}_{\text{transcendental part}},$$

where $g_0, g_1, \ldots, g_n \in F$ and $c_1, \ldots, c_n \in \mathbb{C}$.

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Remark. With the above theorem, Liouville proved that the integrals

$$\int \exp(x^2)\,dx, \quad \int \frac{1}{\log(x)}\,dx, \quad \int \frac{\sin(x)}{x}\,dx, \ldots$$

are not elementary.

Two classical theorems

Liouville-Hardy Theorem. Let $f \in \mathbb{C}(x)$. Then $f \cdot \log(x)$ is elementary integrable over $\mathbb{C}(x)$ if and only if

$$f = \frac{c}{x} + g' \quad \text{for some } c \in \mathbb{C} \text{ and } g \in \mathbb{C}(x).$$

Liouville's Theorem. Let $f, g \in \mathbb{C}(x)$. Then $f \cdot \exp(g)$ is elementary integrable over $\mathbb{C}(x)$ if and only if

$$f = h' + g'h \quad \text{for some } h \in \mathbb{C}(x).$$

Why $\exp(x^2)$ is not Elementary Integrable?

Let $t = \exp(x^2)$. We prove by contradiction.

Proof. If $\int t\,dx$ is elementary, Liouville's theorem implies that $\exists g_0, \ldots, g_n \in \mathbb{C}(x,t)$ and $c_0, \ldots, c_n \in \mathbb{C}$ s.t.

$$\int t\,dx = g_0 + \sum_{i=1}^{n} c_i \log(g_i) \quad \Leftrightarrow \quad t = g_0' + \sum_{i=1}^{n} c_i \frac{g_i'}{g_i}$$

$$\Downarrow$$

$$t = (ft)' \quad \text{for some } f \in \mathbb{C}(x) \quad \Leftrightarrow \quad 1 = f' + 2xf$$

Claim. The differential equation

$$y(x)' + 2x \cdot y(x) = 1$$

has no rational-function solution!

The irrationality of π

Suppose that $\pi/2 = a/b \in \mathbb{Q}$. Consider

$$I_n(x) = \int_{-1}^{1} (1-z^2)^n \cdot \cos(xz)\, dz \quad (n \in \mathbb{N})$$

Let $J_n(x) := x^{2n+1} I_n(x)$. Then

$$J_n(x) = 2n(2n-1)J_{n-1}(x) - 4n(n-1)x^2 J_{n-2}(x).$$

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where $P_n, Q_n \in \mathbb{Z}[x]$ are of degree $\leq n$. Taking $x = \pi/2$ yields

$$\frac{a^{2n+1}}{n!} I_n(\pi/2) = P_n(\pi/2) b^{2n+1} \in \mathbb{N}.$$

But $0 < I_n(\pi/2) < 2$, which implies

$$\frac{a^{2n+1}}{n!} I_n(\pi/2) \to 0 \quad (\text{as } n \to +\infty).$$

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But $0 < I_n(\pi/2) < 2$, which implies

$$\frac{a^{2n+1}}{n!} I_n(\pi/2) \to 0 \quad (\text{as } n \to +\infty). \quad \text{a contradiction!}$$

Stability in dynamical systems

A (discrete) dynamical system is a pair (X, ϕ) with X being any set and $\phi : X \to X$ a self-map on X.

- Subset of fixed points:

$$\mathrm{Fix}(\phi, X) = \{x \in X \mid \phi(x) = x\}.$$

- Subset of periodic points:

$$\mathrm{Per}(\phi, X) = \{x \in X \mid \phi^n(x) = x \text{ for some } n \in \mathbb{N} \setminus \{0\}\}.$$

- Subset of stable points:

$$\mathrm{Stab}(\phi, X) = \{x \in X \mid \exists \{x_i\}_{i \geq 0} \text{ s.t. } x_0 = x \text{ and } \phi(x_{i+1}) = x_i \text{ for } i \in \mathbb{N}\}.$$

- Subset of attractive points:

$$\mathrm{Attrac}(\phi, X) = \bigcap_{i \in \mathbb{N}} \phi^i(X).$$

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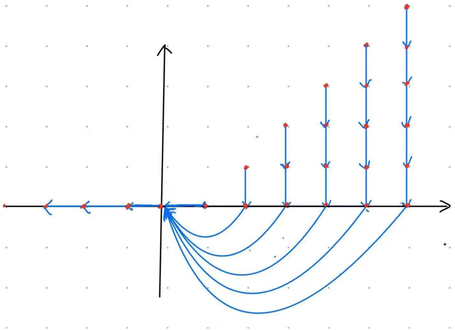
$$\mathrm{Fix}(\phi, X) \subseteq \mathrm{Per}(\phi, X) \subseteq \mathrm{Stab}(\phi, X) \subseteq \mathrm{Attrac}(\phi, X).$$

Godelle's example



Example. Let $X = \{(i,j) \in \mathbb{Z}^2 \mid 0 \le j \le \max\{i-1, 0\}\}$ and $\phi : X \to X$ be such that

$$\phi((i,j)) = (i, j-1) \text{ if } j > 0 \text{ and } \phi((i,0)) = (\min i-1, 0, 0).$$

Then $\mathrm{Stab}(\phi, X) = \emptyset$ and $\mathrm{Attrac}(\phi, X) = \{(i,0) \mid i \le 0\}$.

Stability in differential fields

Idea. Viewing a differential field (K,D) as a dynamical system.

$$D(f+g) = D(f) + D(g) \quad \text{and} \quad D(fg) = gD(f) + fD(g).$$

Definition. $C_K := \{c \in K \mid D(c) = 0\}$ is called the constant subfield of (K,D).

Remark. K is a C_K-vector space and $D: K \to K$ is C_K-linear.

Proposition. Let (K,D) be a differential field of char. zero. Then

$$\mathrm{Stab}(D,K) = \mathrm{Attrac}(D,K).$$

Stability Problem. Given $f \in K$, decide whether f is stable or not, i.e., for all $i \in \mathbb{N}$, $f = D^i(g_i)$ for some $g_i \in K$.

Structure theorem

Lemma. Let (K, D) be a differential field with $D(x) = 1$ and $f \in K$. Then

(i) $f = D^n(g)$ for some $g \in K$ iff for any i with $0 \leq i \leq n-1$, $\exists h_i \in K$ s.t. $x^i f = D(h_i)$.

(ii) f is stable iff for all $i \in \mathbb{N}$, $x^i f = D(g_i)$ for some $g_i \in K$.

Theorem. Let (K, D) be a differential field with $D(x) = 1$. Then Stab(D, K) forms a differential $C_K[x]$-module.

Problem. Is Stab(D, K) always a free $C_K[x]$-module?

Example. $\exp(c \cdot x)$ is stable, so are

$$x^n \exp(c \cdot x), \quad x^n \sin(c \cdot x), \quad x^n \cos(c \cdot x), \quad \ldots$$

Stable elementary functions

Let $\mathscr{E}_{\mathbb{C}(x)}$ be the field of all elementary functions over $\mathbb{C}(x)$.

Theorem. Let $D = d/dx$ and $f, g \in \mathbb{C}(x)$ with $g \notin \mathbb{C}$. Then

(i) f is always stable in $(\mathscr{E}_{\mathbb{C}(x)}, D)$.

(ii) f is stable in $(\mathbb{C}(x), D)$ iff $f \in \mathbb{C}[x]$.

(iii) $f \cdot \log(x)$ is stable in $(\mathscr{E}_{\mathbb{C}(x)}, D)$ iff $f \in \mathbb{C}[x, x^{-1}]$.

(iv) $f \cdot \exp(g)$ is stable in $(\mathscr{E}_{\mathbb{C}(x)}, D)$ iff $f \in \mathbb{C}[x]$ and $g = ax + b$ with $a, b \in \mathbb{C}$ with $a \neq 0$.

Examples. Stable elementary functions: $f(x) \in \mathbb{C}(x)$, $\exp(ax+b)$,

$$\log(f(x)), \quad \sin(x), \quad \cos(x), \quad \arcsin(x) \quad \arccos(x), \quad \arctan(x), \ldots$$

Non-stable elementary functions: $\tan(x)$, $\cot(x)$, $\sec(x)$, $\csc(x)$, $\ldots$

D-finite power series and exact integration

Definition. $f(x) \in \mathbb{C}[[x]]$ is D-finite over $\mathbb{C}(x)$ if $\exists\, L = \sum_{i=0}^{r} \ell_i \cdot D_x^i$ in $\mathbb{C}(x)\langle D_x \rangle$ with $\ell_r \neq 0$ s.t. $L(f) = 0$, equivalently

$$\dim_{\mathbb{C}(x)} \left(\mathrm{span}_{\mathbb{C}(x)} \{ D_x^i(f) \mid i \in \mathbb{N} \} \right) < +\infty.$$

If L is monic and of minimal order r, then call L the minimal annihilator for f and call r the order of f, denoted by $\mathrm{ord}(f)$.

Remark. In general, the formal integral int(f):= $\int f(x)dx$ has the minimal annihilator of order $\mathrm{ord}(f) + 1$.

Exact Integration. In 1997, Abramov and van Hoeij gave an algorithm to decide whether $\int f(x)dx$ has an annihilator of the same order as that of f.

Stable D-finite power series

Let $f(x) \in \mathbb{C}[[x]]$ be a D-finite power series.

Definition. $f(x)$ is stable if $\exists \{g_i\}_{i\in\mathbb{N}} \in \mathbb{C}[[x]]$ s.t. $g_0 = f$ and

$$g_i = D_x(g_{i+1}) \text{ and } \operatorname{ord}(g_i) = \operatorname{ord}(f) \quad \text{for all } i \in \mathbb{N}.$$

$f(x)$ is eventually stable if $\exists\, m \in \mathbb{N}$ s.t. $\operatorname{int}^m(f)$ is stable.

Theorem. Any D-finite power series is eventually stable.

Example. The Airy function $\operatorname{Ai}(x)$ satisfies

$$y''(x) = xy(x).$$

By Abramov-van Hoeij's algorithm, we have $\operatorname{Ai}(x)$ is not stable, but is eventually stable with $\operatorname{ord}(\operatorname{int}^m(\operatorname{Ai}(x))) = 3$ for all $m \geq 2$.

Difference Algebra

Difference Ring and Difference Field. Let R be an integral domain and σ be an automorphism of R. The pair (R, σ) is called a difference ring. If R is a field, it is then called a difference field. Let Δ be the difference operator such that $\Delta(r) = \sigma(r) - r$ for $r \in R$.

Examples.

- Polynomial ring: $(\mathbb{C}[x], \sigma)$ with $\sigma(P(x)) = P(x+1)$ for any $P \in \mathbb{C}[x]$.
- Rational-function field: $(\mathbb{C}(x), \sigma)$ with $\sigma(f(x)) = f(x+1)$ for any $f \in \mathbb{C}(x)$.
- Let S be the ring of sequences $a(n) : \mathbb{N} \to \mathbb{C}$ and I be the ideal of sequences with only finitely many terms are nonzero. Let $R = S/I$ with $\sigma(a(n)+I) = a(n+1)+I$.

Stability in difference fields

Idea. Viewing a difference field (K, Δ) as a dynamical system.

$$\Delta(f+g) = \Delta(f) + \Delta(g) \quad \text{and} \quad \Delta(fg) = \sigma(f)\Delta(g) + g\Delta(f).$$

Definition. $C_K := \{c \in K \mid \Delta(c) = 0\}$ is called the constant subfield of (K, σ).

Remark. K is a C_K-vector space and $\Delta : K \to K$ is C_K-linear.

Proposition. Let (K, Δ) be a difference field of char. zero. Then

$$\text{Stab}(\Delta, K) = \text{Attrac}(\Delta, K).$$

Stability Problem. Given $f \in K$, decide whether f is stable or not, i.e., for all $i \in \mathbb{N}$, $f = \Delta^i(g_i)$ for some $g_i \in K$.

Structure theorem

Theorem. Let (K,Δ) be a difference field with $\Delta(x) = 1$. Then $\mathrm{Stab}(\Delta, K)$ forms a $C_K[x]$-module.

Examples.

$$n^2, \quad 2^n, \quad \frac{\binom{2n}{n}}{4^n}, \quad \ldots$$

are stable sequences.

$$\sum_{n_s=0}^{n} \sum_{n_{s-1}=0}^{n_s} \cdots \sum_{n_1=0}^{n_2} \frac{\binom{2n_1}{n_1}}{4^{n_1}} = \frac{(2n+2s-1)!!}{(2n-1)!!(2s-1)!!} \frac{\binom{2n}{n}}{4^n} = \frac{\binom{2n+2s}{2s}}{\binom{n+s}{s}} \frac{\binom{2n}{n}}{4^n}$$

Problem. Is $\mathrm{Stab}(\Delta, K)$ always a free $C_K[x]$-module?

Stable hypergeometric sequences

Definition. A sequence $T : \mathbb{N} \to C$ is hypergeometric over $C(n)$ if

$$\frac{T(n+1)}{T(n)} \in C(n).$$

Two hypergeometric sequences T_1, T_2 are similar if $T_1/T_2 \in C(n)$.

Remark. The set $H_T := \{f \cdot T \mid f \in C(n)\}$ froms a vector space over C that is closed under Δ. Moreover, $\dim_C(\ker(\Delta)) = 1$. Then

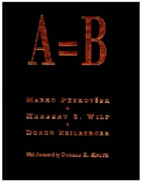
$$\mathrm{Stab}(\Delta, H_T) = \mathrm{Attrac}(\Delta, H_T).$$

Problem. Classifying all possible stable hypergeometric $T(n)$ over $C(n)$, i.e., for all $i \in \mathbb{N}$, $T = \Delta^i(G_i)$ for some hypergeometric G_i.

Stable hypergeometric sequences

An identity from the book **A=B**:

$$\sum_{n_s=0}^{n} \sum_{n_{s-1}=0}^{n_s} \cdots \sum_{n_1=0}^{n_2} \frac{\binom{2n_1}{n_1}}{4^{n_1}} = \frac{(2n+2s-1)!!}{(2n-1)!!(2s-1)!!} \frac{\binom{2n}{n}}{4^n} = \frac{\binom{2n+2s}{2s}}{\binom{n+s}{s}} \frac{\binom{2n}{n}}{4^n}$$



Problem. Classifying iteratively summable (stable) hypergeometric sequences.

Classification Theorem. A hypergeometric $H(k)$ is stable iff $H(k)$ is

- Exp-polynomial: $p(k) \cdot \alpha^k$ with $p \in \mathbb{C}[k], \alpha \in \mathbb{C} \setminus \{0\}$ or
- Gamma-polynomial: $p(k) \cdot \frac{\Gamma(k+\alpha)}{\Gamma(k+\beta)}$ with $p \in \mathbb{C}[k], \alpha, \beta \in \mathbb{C}$ and $\alpha - \beta \notin \mathbb{Z}$.

Accurate Summation

Definition. Let $a(n)$ be a P-recursive sequence

$$p_d \cdot a(n+d) + p_{d-1} \cdot a(n+d-1) + \cdots + p_0 \cdot a(n) = 0.$$

If d is minimal, then call d the order of $a(n)$, denoted by $\text{ord}(a(n))$.

Remark. In general, the indefinite sum

$$s(n) = a(1) + a(2) + \cdots + a(n),$$

satisfies a linear recurrence of order $\text{ord}(a) + 1$.

Accurate Summation. In 1997, Abramov and van Hoeij gave an algorithm to decide whether $\text{ord}(s(n)) = \text{ord}(a(n))$.

Stable P-recursive sequences

Let $a(n)$ be a P-recursive sequence.

Definition. $a(n)$ is stable if $\exists\ \{g_i\}_{i\in\mathbb{N}} \in S/I$ s.t. $g_0 = a(n)$ and

$$g_i = \Delta(g_{i+1}) \text{ and } \mathrm{ord}(g_i) = \mathrm{ord}(a(n)) \quad \text{for all } i \in \mathbb{N}.$$

$a(n)$ is eventually stable if $\exists\ m \in \mathbb{N}$ s.t. $\sum^m (a(n))$ is stable.

Example. Let $a(n) = 1/n$ and $H(n) = \sum_{i=1}^{n-1} a(i)$ with $\Delta(H) = a$. We have

$$(n+1)a(n+1) - na(n) = 0.$$

$$(n+1)H(n+2) - (2n+1)H(n+1) + nH(n) = 0.$$

By Abramov-van Hoeij's algorithm, we have $a(n)$ is not stable, but is eventually stable with $\mathrm{ord}(\sum^m a(n)) = 2$ for all $m \geq 2$.

Theorem. Any P-recursive sequence is eventually stable.

Thank You!